NSG-3186 9/78
R-O 12-78
2-79
4-79
7-79

ME-GL-78-1

TWO-DIMENSIONAL DUCT FLOW CODE

by

Mahesh S. Greywall

(NASA-CR-162133) TWO-DIMENSIONAL DUCT FLOW CODE (Wichita State Univ.) 30 p
N79-79010
Unclas
00/34 34131

Department of Mechanical Engineering
Wichita State University
Wichita, Kansas 67208

Mahesh S. Greywall
Department of Mechanical Engineering
Wichita State University
Wichita, Kansas 67208

ABSTRACT

A computational method to calculate momentum and energy transport in two-dimensional viscous compressible duct flows is presented. The flow in the duct is partitioned into finite streams. The difference equations are then obtained by applying momentum and energy conservation principles directly to the individual streams. The method is applicable to laminar and turbulent flows.

I. INTRODUCTION

Two-dimensional effects in duct flows, such as the growth of momentum and thermal boundary layers, have been studied for a long time by separating the flow into a core flow and a boundary layer [1]. Two-dimensional effects associated with sudden changes in duct shape in incompressible flows have been successfully computed using vorticity transport equation [2]. In high temperature compressible flows such as those encountered in magnetohydrodynamic (MHD) channels separation of flow into a uniform core and a boundary layer is difficult except at the very beginning of the channel. The difficulties arise from (1) large temperature differences between the wall and the core flow which lead to temperature gradients within the core and (2) shrinking size of the core as the flow proceeds down the channel. Also in calculating performance of diffusers with high inlet blockage factor, as in MHD diffusers, one cannot separate the flow into a core and a boundary layer. Computation of such flows have been recently carried out mostly by the methods based on the work of Patankar and Spalding [3,4]. Some MHD channel calculations have also been carried out using Fluid-in-Cell Method [5,6].

We have in this article presented a new approach to compute internal flows without separating the flow into a core and a boundary layer. A novel feature of this approach is the method by which the difference equations are obtained. In most of the approaches one starts with the differential equations, such as energy equation, Navier-Stokes equations, which are obtained from the conservation principles. These equations either in

their primitive forms or in transformed forms, such as vorticity equation or the equations obtained by von Mises transformation as used in Patankar-Spalding method [3], are then differenced to obtain the finite difference equations needed to compute the flow. In the present approach the flow in the duct is portioned into finite streams and the finite difference equations are then obtained by applying conservation principles directly to the individual streams. The direct relationship of the finite difference equations to the conservation principles makes it easier to introduce approximations allowed by the flow being simulated and also to incorporate any special features of the flow.

In section II we have presented the finite difference equations based on the new approach and in section III discussed their solution techniques. The discussion is limited to two-dimensional flows. The application of the present approach to compute three-dimensional effects is straightforward and is indicated at the end of section III. In this paper we have tried to present the new approach in broad and general terms to make its application easier to specific problems requiring individual handling. We have used this approach to compute two-dimensional viscous compressible turbulent flows in hydrogen-oxygen combustion driven MHD ducts [7] and are currently using it to calculate diffuser performance with high inlet blockage factors.

II. FINITE DIFFERENCE EQUATIONS

Consider two dimensional duct flow as shown in Fig. 1. The direction x is taken along the flow and z, measured from the duct wall, normal to it. For circular ducts z represents the radial distance from the wall. For rectangular ducts the variation of flow properties normal to z and x are neglected. We partition the total flow in the duct into a sequence of finite streams. To accomplish this partitioning we draw a series of streamlines 1,2,....,N as shown in Fig. 1. Streamline 1 is taken along the wall and streamline N represents the duct center or the symmetry line. The spacing between any two consecutive streamlines is determined by the desired resolution of the local gradients normal to the flow. We define streamline $j+1/2$ as the streamline passing through the middle of j and $j+1$. Streamline $j-1/2$ is defined similarly. The stream j, in general, is now defined to be the flow bounded by streamlines $j-1/2$ and $j+1/2$. The shape of the stream, of course, depends on the shape of the duct: in rectangular ducts the streams will be plane layers and in circular ducts the streams will be cylindrical shells. Stream 2 is defined to be the flow between the wall and streamline $2+1/2$. Stream N is defined to be the flow between the streamline $N-1/2$ and the center or symmetry line N.

We introduce the following nomenclature:

ψ_j^+; mass flow rate between the streamline j and $j+1/2$.

ψ_j^-; mass flow rate between the streamline j and $j-1/2$.

Note $\psi_j^+ + \psi_j^-$ represents the mass flow rate through stream j. For calculations of ψ_j's see Appendix A.

$u_j(x)$, $T_j(x)$, $\rho_j(x)$, $h_j(x)$, $C_j(x)$; velocity, temperature, density, enthalpy and constant pressure specific heat along the streamline j.

$z_j(x)$; distance of streamline j from the wall.

A_j^x; cross-sectional area of stream j (see Appendix A).

$\mu_{j+1/2}(x)$, $k_{j+1/2}(x)$; effective viscosity and thermal conductivity (including turbulent contribution when the flow is turbulent) along streamline $j+1/2$. These are the coefficients that are later used to calculate momentum and energy transfer between the stream j and $j+1$ across the interface at $j+1/2$.

Other nomenclature is introduced as we go along.

The difference equations to calculate the flow properties are now obtained by applying conservation principles directly to the individual streams. We assume that the streamlines are chosen sufficiently close so that the variation of the flow properties between any two successive streamlines j and $j+1$ can be approximated as linear. Note the flow properties u_j, T_j, etc., though not their gradients, are continuous across these streamlines. However, across the interface of the streams, i.e. across streamlines $j+1/2$, the gradients are also continuous thus determining, unambiguously, momentum and energy transfer between the adjoining streams.

Applying momentum balance to stream j between x and $x+\Delta x$, i.e. applying the momentum balance to the control volume bounded by x, $x+\Delta x$, $j-1/2$, and $j+1/2$, we obtain,

$$\text{Momentum Flux Out}\Big|_{\text{at } x+\Delta x} - \text{Momentum Flux In}\Big|_{\text{at } x}$$

$$= \text{Pressure Forces} + \text{Viscous Forces} + \text{Body Forces} \qquad (1)$$

Estimation of the various quantities in Eq. (1) allows a great deal of flexibility to accommodate the desired accuracy and any special features of the given flow. We here present one such approximation for stream j, (streams 2 and N require special treatment and are considered later separately).

$$\left[\psi_j^+\left(\frac{3u_j^+ + u_{j+1}^+}{4}\right) + \psi_j^-\left(\frac{3u_j^+ + u_{j-1}^+}{4}\right)\right]$$

$$- \left[\psi_j^+\left(\frac{3u_j + u_{j+1}}{4}\right) + \psi_j^-\left(\frac{3u_j + u_{j-1}}{4}\right)\right]$$

$$= -\Delta p A_j^x + \frac{1}{2}\left[V_{j+1/2}^+(u_{j+1}^+ - u_j^+) + V_{j+1/2}(u_{j+1} - u_j)\right]$$

$$- \frac{1}{2}\left[V_{j-1/2}^+(u_j^+ - u_{j-1}^+) + V_{j-1/2}(u_j - u_{j-1})\right] + F_j \qquad (2)$$

On the left hand side of Eq. (2) the expressions involving ψ_j^+ represents the momentum flux between j and $j + 1/2$ and those involving ψ_j^- the momentum flux between $j - 1/2$ and j. The superscript + indicates the unknown velocities at $x + \Delta x$, Δp the pressure difference between $x + \Delta x$ and x, and F_j the sum of body forces on the control volume. Also we have introduced

$$V_{j+1/2} = \frac{\mu_{j+1/2}}{z_{j+1} - z_j} A_{j+1/2}^s, \qquad V_{j-1/2} = \frac{\mu_{j-1/2}}{z_j - z_{j-1}} A_{j-1/2}^s \qquad (3)$$

where $A_{j+1/2}^s$ and $A_{j-1/2}^s$ represent the upper and lower surface areas of the control volume (see Appendix A for calculation of A^s's). Terms $V_{j+1/2}^+$ and $V_{j-1/2}^+$ are defined similarly with μ's and z's evaluated at $x + \Delta x$. Note we have estimated the viscous forces on a given surface by taking the average

of those at x and $x+\Delta x$. Rewritting Eq. (2) with all the unknown velocities collected on the right hand side we obtain,

$$-u_{j+1}^{+}\left[-\frac{\psi_j^{+}}{4}+\frac{1}{2}V_{j+1/2}^{+}\right]+u_j^{+}\left[\frac{3}{4}\psi_j^{+}+\frac{3}{4}\psi_j^{-}+\frac{1}{2}V_{j+1/2}^{+}+\frac{1}{2}V_{j-1/2}^{+}\right]$$

$$-u_{j-1}^{+}\left[-\frac{\psi_j^{-}}{4}+\frac{1}{2}V_{j-1/2}^{+}\right]$$

$$=-\Delta pA_j^{x}+u_{j+1}\left[\frac{\psi_j^{+}}{4}+\frac{1}{2}V_{j+1/2}\right]+u_j\left[\frac{3}{4}\psi_j^{+}+\frac{3}{4}\psi_j^{-}-\frac{1}{2}V_{j+1/2}-\frac{1}{2}V_{j-1/2}\right]$$

$$+u_{j-1}\left[\frac{\psi_j^{-}}{4}+\frac{1}{2}V_{j-1/2}\right]+F_j$$

$$j=3,4,\ldots,N-1 \qquad (4)$$

Similarly, application of momentum balance to stream N, after some rearrangement, yields

$$u_N^{+}\left[\frac{3}{4}\psi_N^{-}+\frac{1}{2}V_{N-1/2}^{+}\right]-u_{N-1}^{+}\left[-\frac{\psi_N^{-}}{4}+\frac{1}{2}V_{N-1/2}^{+}\right]$$

$$=-\Delta pA_N^{x}+u_N\left[\frac{3}{4}\psi_N^{-}-\frac{1}{2}V_{N-1/2}\right]+u_{N-1}\left[\frac{\psi_N^{-}}{4}+\frac{1}{2}V_{N-1/2}\right]+F_N \qquad (5)$$

We note that Eq. (4) will yield Eq. (5) for j=N if we were to set ψ_N^{+}, $V_{N+1/2}^{+}$, $V_{N+1/2}$, and u_{N+1} equal to zero. Thus by initializing these quantities equal to zero at the beginning of a computer program stream N can be treated by the general formula for stream j.

Next we will treat Stream 2. This stream requires considerably more computational finesse. The sources of the difficulty are (1) occurrence of steep

gradients normal to the flow near the wall and (2) reckoning of wall surface roughness for turbulent flows. For laminar flows the wall surface roughness does not effect the flow and the gradients are not as steep and the flow can be simulated by taking streams sufficiently thin next to the wall. For turbulent flows we let go of the assumption that the velocity varies linearly between the wall and streamline 2. The momentum flux through the lower half of the stream 2 is estimated as some α times $\psi_2^- u_2$ where α takes into account the nonlinearity of the velocity distribution next to the wall. Also the viscous force at the lower face of stream 2, that is at the wall, is calculated from the Reynold's number along streamline 2 taking into account the wall surface roughness. Proceeding accordingly, we obtain by applying momentum balance to stream 2

$$\left[\psi_2^+\left(\frac{3u_2^+ + u_3^+}{4}\right) + \psi_2^- \alpha u_2^+\right] - \left[\psi_2^+\left(\frac{3u_2 + u_3}{4}\right) + \psi_2^- \alpha u_2\right]$$

$$= -\Delta p A_2^x + \frac{1}{2}\left[V_{2+1/2}^+(u_3^+ - u_2^+) + V_{2+1/2}(u_3 - u_2)\right]$$

$$- \left[V_1^+ u_2^+ + V_1 u_2\right] + F_2 \qquad (6)$$

Coefficients $V_{2+1/2}^+$ and $V_{2+1/2}$ are given by the general expression for $V_{j+1/2}$ presented earlier. For programming ease viscous forces along the wall have been expressed as $V_1 u_2$ similar to viscous forces at the other interfaces. However V_1 is defined differently from other V's. Estimations of V_1 and α for turbulent flows are discussed in Appendix B. When the flow is laminar Eq. (6) can be used for the momentum balance for stream 2 with α set equal to one half and V_1 determined by

$$V_1 = \frac{\mu_1}{z_2} A_1^s \qquad \text{(Laminar Flow)} \qquad (7)$$

where the A_1^s represents the area of the lower surface of stream 2 and μ_1 the viscosity at the wall (V_1^+ is determined similarly). Rearranging Eq. (6) we obtain

$$-u_3^+\left[-\frac{\psi_2^+}{4} + V_{2+1/2}^+\right] + u_2^+\left[\frac{3}{4}\psi_2^+ + \alpha\psi_2^- + V_{2+1/2}^+ + V_1^+\right]$$

$$= -\Delta p A_2^x + u_3\left[\frac{\psi_2^+}{4} + V_{2+1/2}\right] + u_2\left[\frac{3}{4}\psi_2^+ + \alpha\psi_2^- - V_{2+1/2} - V_1\right] + F_2 \qquad (8)$$

Equations (4), (5), and (8) provide N-1 equations to solve for the N-1 unknown velocities $u_2^+, u_3^+, \ldots, u_N^+$.

Equations for the unknown enthalpies h_j^+ at $x + \Delta x$ are obtained by applying conservation of energy to the control volume under consideration;

$$\text{Energy Flux Out}\Big|_{\text{at } x+\Delta x} - \text{Energy Flux In}\Big|_{\text{at } x} =$$

$$\text{Heat Added By Conduction} + \text{Work Done by Viscous Forces} +$$

$$Q_j \text{ (Internal Heat Generated)} + W_j \text{ (Work Done by the Body Forces)} \qquad (9)$$

The different terms in Eq. (9) can be estimated with varying degree of accuracy (and the accompanying complexity) to reflect their relative importance for the flow being simulated. One estimate for stream j is as follows:

$$\text{Energy Flux In} = \frac{\psi_j^+}{4}\left[3h_j + h_{j+1} + \frac{1}{2}(3u_j^2 + u_{j+1}^2)\right]$$

$$+ \frac{\psi_j^-}{4}\left[3h_j + h_{j-1} + \frac{1}{2}(3u_j^2 + u_{j-1}^2)\right]$$

Expression for energy flux out is obtained from the expression for the energy flux in by adding superscript + to h's and u's.

Heat added by conduction across the upper surface of the control volume

$$= \frac{1}{2} H^{+}_{j+1/2}(h^{+}_{j+1} - h^{+}_{j}) + \frac{1}{2} H_{j+1/2}(h_{j+1} - h_j)$$

where we have introduced

$$H^{+}_{j+1/2} = \frac{k^{+}_{j+1/2}}{C^{+}_{j}(z^{+}_{j+1} - z^{+}_{j})} A^{s}_{j+1/2} \quad \text{and} \quad H_{j+1/2} = \frac{k_{j+1/2}}{C_j(z_{j+1} - z_j)} A^{s}_{j+1/2} \tag{10}$$

The H terms can be expressed in terms of V by using Pr, the Prandtl number. Thus we can write

$$H_{j+1/2} = \frac{V_{j+1/2}}{Pr_{j+1/2}} \quad \text{where} \quad Pr_{j+1/2} = \frac{\mu_{j+1/2} C_j}{k_{j+1/2}} \tag{11}$$

Work done by the viscous forces on the upper surface

$$= \frac{1}{2}\left[V^{+}_{j+1/2}(u^{+}_{j+1} - u^{+}_{j})\left(\frac{u^{+}_{j+1} + u^{+}_{j}}{2}\right) + V_{j+1/2}(u_{j+1} - u_j)\left(\frac{u_{j+1} + u_j}{2}\right)\right]$$

$$= \frac{1}{4}\left[V^{+}_{j+1/2}(u^{+2}_{j+1} - u^{+2}_{j}) + V_{j+1/2}(u^{2}_{j+1} - u^{2}_{j})\right]$$

Expressions for the heat transferred and the work done across the lower surface can be written down from the corresponding one's for the upper surface by replacing $j+1$ by j, j by $j-1$, and $j+1/2$ by $j-1/2$. Substituting the various estimates into Eq. (7) and rearranging to get all the unknown h^{+}_{j}'s on the left hand side of the equation we get,

$$-h_{j+1}^{+}\left[-\frac{\psi_j^{+}}{4}+\frac{1}{2}H_{j+1/2}^{+}\right]+h_j^{+}\left[\frac{3}{4}\psi_j^{+}+\frac{3}{4}\psi_j^{-}+\frac{1}{2}H_{j+1/2}^{+}+\frac{1}{2}H_{j-1/2}^{+}\right]$$

$$-h_{j-1}^{+}\left[-\frac{\psi_j^{-}}{4}+\frac{1}{2}H_{j-1/2}^{+}\right]$$

$$=h_{j+1}\left[\frac{\psi_j^{+}}{4}+\frac{1}{2}H_{j+1/2}\right]+h_j\left[\frac{3}{4}\psi_j^{+}+\frac{3}{4}\psi_j^{-}-\frac{1}{2}H_{j+1/2}-\frac{1}{2}H_{j-1/2}\right]$$

$$+h_{j-1}\left[\frac{\psi_j^{-}}{4}+\frac{1}{2}H_{j-1/2}\right]+\frac{u_{j+1}^{+2}}{2}\left[-\frac{\psi_j^{+}}{4}+V_{j+1/2}^{+}\right]$$

$$-\frac{u_j^{+2}}{2}\left[\frac{3}{4}\psi_j^{+}+\frac{3}{4}\psi_j^{-}+V_{j+1/2}^{+}+V_{j-1/2}^{-}\right]+\frac{u_{j-1}^{+2}}{2}\left[-\frac{\psi_j^{-}}{4}+V_{j-1/2}^{+}\right]$$

$$+\frac{u_{j+1}^{2}}{2}\left[\frac{\psi_j^{+}}{4}+V_{j+1/2}\right]+\frac{u_j^{2}}{2}\left[\frac{3}{4}\psi_j^{+}+\frac{3}{4}\psi_j^{-}-V_{j+1/2}-V_{j-1/2}\right]$$

$$+\frac{u_{j-1}^{2}}{2}\left[\frac{\psi_j^{-}}{4}+V_{j-1/2}\right]+Q_j+W_j \qquad (12)$$

for $j = 3,4,\ldots,N-1$

The equation that results on applying the energy balance to stream N can be obtained from Eq. (12) by letting j equal to N and setting ψ_N^{+}, h_{n+1}^{+}, h_{n+1}, u_{N+1}^{+}, u_{N+1}, $H_{N+1/2}^{+}$, $H_{N+1/2}$, $V_{N+1/2}^{+}$, and $V_{N+1/2}$ equal to zero.

Next, following the procedure outlined for the momentum balance, we apply energy balance to stream 2 to obtain,

$$\frac{\psi_2^+}{4}\left[(3h_2^+ + h_3^+) + \frac{1}{2}(3u_2^{+2} + u_3^{+2})\right]$$

$$-\frac{\psi_2^+}{4}\left[(3h_2 + h_3) + \frac{1}{2}(3u_2^2 + u_3^2)\right]$$

$$+\psi_2^-\left[\beta(h_2^+ - h_w^+) + \delta\frac{1}{2}u_2^{+2}\right] - \psi_2^-\left[\beta(h_2 - h_w) + \frac{1}{2}\delta u_2^2\right]$$

$$= \frac{1}{2}\left[H_{2+1/2}^+(h_3^+ - h_2^+) + H_{2+1/2}(h_3 - h_2)\right] - q_w$$

$$+\frac{1}{4}\left[V_{2+1/2}^+(u_3^{+2} - u_2^{+2}) + V_{2+1/2}(u_3^2 - u_2^2)\right] + Q_2 + W_2 \qquad (13)$$

In Eq. (13) β and δ are correction factors to take care of nonlinear variation between wall and the streamline 2 when the flow is turbulent and are further discussed in Appendix B. Also in Eq. (13) h_w represents the enthalpy evaluated at the wall temperature and q_w the heat transfer to the wall. For flows with wall temperature specified q_w can be written as

$$q_w = \frac{1}{2}\left[H_1^+(h_2^+ - h_w^+) + H_1(h_2 - h_w)\right] \qquad (14)$$

where H_1 is calculated from V_1 via the relation given in Eq. (11). When q_w is specified Eq. (14) is used to eliminate h_w and h_w^+ appearing on the left hand side of Eq. (13). Equations (12), (13) and (14) provide N equations for the N unknowns h_N, h_{N-1},...,h_2 and either one of h_w or q_w.

The proceeding equations for the calculations of u_j^+'s and h_j^+'s are to be supplemented with equations given in Appendices A and B to calculate the duct width at $x + \Delta x$ in terms of u_j^+'s and ρ_j^+'s.

In summary, Eqs. (4), (5), (8), (12), (13), (14), (B-2), (A-13) [or (B-3), (A-14), and (A-15)], along with the thermodynamic relations interrelating h, ρ, T, and p are our basic set of equations to calculate the two dimensional duct flow.

III. SOLUTION TECHNIQUE

Overall methodology adopted to solve the duct flow equations will depend upon the given design specifications, e.g. pressure gradient is specified or variation of duct width is specified, etc. In any case one has to solve the momentum and energy difference equations given in section II. The set of simultaneous equations to solve for new u_j^+, as also the set to solve for new h_j^+, are of the general form

$$-a_j X_{j+1} + b_j X_j - c_j X_{j-1} = d_j, \tag{15}$$

where X_j stands for the unknowns u_j^+ or h_j^+. Coefficients a_j, b_j and c_j due to their dependence on V_j^+ or H_j^+ are indirectly functions of X_j. Thus Eq. (15) is nonlinear. If we had approximated the viscous forces and the heat transfer across stream interfaces by their values at x rather than by their averages between x and $x + \Delta x$ coefficients a_j, b_j and c_j would have been independent of X_j. In that case Eq. (15) will constitute a set of linear tri-diagonal equations and can be solved directly by the standard algorithm to solve such equations. In the present case these equations are solved by iteration.

At first we calculate a_j, b_j, and c_j for the momentum difference equations setting V^+ equal to V and solve for new u_j^+ using the algorithm to solve directly tri-diagonal equations. With these new u_j^+ and H^+ set equal to H we calculated a_j, b_j, c_j and d_j for the energy difference equations and solve for h_j^+. Next we calculate z_j^+ using newly calculated flow properties at $x + \Delta x$. The procedure is then repeated this time around calculating V^+ and H^+ using the newly calculated values of u_j^+, z_j^+, etc. The iteration

is continued until the desired convergence is achieved. The number of required iterations increases in regions of rapidly changing temperature and velocity profiles. These are also the regions where had we calculated a_j, b_j and c_j from V and H alone (instead V, V^+ and H, H^+) the error introduced by such approximation would have been the greatest.

When the pressure gradient is specified the z_j^+'s calculated at the end of the last iteration give the channel width at $x + \Delta x$. When the channel width is specified iteration on pressure gradient is required. The iteration in the pressure gradient can be performed either over and above the iteration on V^+ and H^+ or combined into the V^+-H^+ iteration loop. In the latter case at the end of each iteration when new z_j^+ and thus the channel width at $x + \Delta x$ is calculated the assumed pressure gradient is readjusted.

We conclude this section by indicating how the present approach can be used to calculate three-dimensional flows. In three-dimensional case one will partition the flow in the duct into three-dimensional streams. In rectangular channels the streams now instead of being plane layers as in two-dimensional case will be rectangular bars. The difference equations to calculate the flow can now be written down as in the two dimensional case except now when calculating viscous forces acting upon the control volume and the heat conducted into it one will have to take into account four neighboring streams. The resulting equations now will be pentadiagonal and will require more computational time to solve.

APPENDIX A

In this appendix we present formulae for calculating ψ_j^+, ψ_j^-, A_j^x, A_j^s, and z_j for the rectangular and cylindrical ducts. These are the formulae we have used in our computational works and can be easily modified to fit the requirements of other problems. For cylindrical ducts we introduce r_j, the distance of streamline j from the centerline. For duct of radius R_0, r_j and z_j are related by

$$r_j = R_0 - z_j \tag{A-1}$$

For rectangular ducts the formulae are for a duct of unit width. (Formulae are presented in forms to reveal their origin. They can be further contracted or simplified for computational efficiency.)

MASS FLOW RATES

(a) Rectangular Ducts

$$\psi_j^+ = \left(\frac{3\rho_j u_j + \rho_{j+1} u_{j+1}}{4}\right)\left(\frac{z_{j+1} + z_j}{2} - z_j\right)$$

$$j = 2,3,\ldots,N-1 \tag{A-2}$$

$$\psi_j^- = \left(\frac{3\rho_j u_j + \rho_{j-1} u_{j-1}}{4}\right)\left(z_j - \frac{z_j + z_{j-1}}{2}\right)$$

$$j = 3,4,\ldots,N \tag{A-3}$$

(b) Cylindrical Ducts

$$\psi_j^+ = \left(\frac{3\rho_j u_j + \rho_{j+1} u_{j+1}}{4}\right)\pi\left[r_j^2 - \left(\frac{r_j + r_{j+1}}{2}\right)^2\right]$$

$$j = 2,3,\ldots,N-1 \qquad \text{(A-4)}$$

$$\psi_j^- = \left(\frac{3\rho_j u_j + \rho_{j-1} u_{j-1}}{4}\right)\pi\left[\left(\frac{r_j + r_{j-1}}{2}\right)^2 - r_j^2\right]$$

$$j = 3,4,\ldots,N-1 \qquad \text{(A-5)}$$

$$\psi_N^- = \left(\frac{3\rho_N u_N + \rho_{N-1} u_{N-1}}{4}\right)\pi r_{N-1}^2 \qquad \text{(A-6)}$$

Calculations of ψ_2^- are discussed in Appendix B.

CROSS-SECTIONAL AREAS

(a) Rectangular Ducts

$$A_2^x = \frac{z_3 + z_2}{2} \qquad \text{(A-7)}$$

$$A_j^x = \frac{z_{j+1} - z_{j-1}}{2} \qquad \text{(A-8)}$$

$$j = 3,4,\ldots,N-1$$

$$A_N^x = \frac{z_N - z_{N-1}}{2} \qquad \text{(A-9)}$$

(b) Cylindrical Ducts

$$A_2^x = \pi\left[r_1^2 - \left(\frac{r_2 + r_3}{2}\right)^2\right] \tag{A-10}$$

$$A_j^x = \pi\left[\left(\frac{r_j + r_{j-1}}{2}\right)^2 - \left(\frac{r_j + r_{j+1}}{2}\right)^2\right] \tag{A-11}$$

$$j = 3,4,\ldots,N-1$$

$$A_N^x = \pi r_{N-1}^2 \tag{A-12}$$

z_j AND r_j TERMS

When the ρ's and u's have been calculated at a new position along the duct the z's and r's at the new position are then calculated by the following formulae,

(a) Rectangular Ducts

After z_2 has been calculated as discussed in Appendix B we calculate $z_3, z_4,\ldots,z_N$ by

$$z_j = z_{j-1} + \frac{8\psi_j^-}{3\rho_j u_j + \rho_{j-1}u_{j-1}} \tag{A-13}$$

(b) Circular Ducts

$$r_{N-1} = \left[\frac{4\psi_N^-}{\pi(3\rho_N u_N + \rho_{N-1}u_{N-1})}\right]^{1/2} \tag{A-14}$$

$$r_{j-1} = 2\left[\frac{4\psi_j^-}{\pi(3\rho_j u_j + \rho_{j-1}u_{j-1})} + r_j^2\right]^{1/2} - r_j$$

$$j = N-1,\ldots,3. \qquad \text{(A-15)}$$

Radius r_1 is calculated as discussed in Appendix B.

SURFACE AREAS

(a) Rectangular Ducts

for all interfaces $\qquad A^s_{j+1/2} = \Delta x \qquad \text{(A-16)}$

(b) Circular Duct

for all interfaces $\qquad A^s_{j+1/2} = \Delta x\, \pi\, (r_j + r_{j+1}) \qquad \text{(A-17)}$

APPENDIX B

In this appendix we will discuss how to calculate the various correction factors that take into account the non-linear nature of velocity distribution between streamline 2 and the wall. There are many approaches possible to calculate these correction factors. The discussion that follows presents a method rather than the only one. For flow through circular pipes thickness of stream 2, i.e. $z_2 - z_1$, is assumed so small compared with the duct radius that we can neglect the curvature of stream 2. In evaluating the various integrals we have held ρ constant equal to some $\bar{\rho}$. For incompressible flows $\bar{\rho}$ is, of course, just equal to ρ. For compressible flows $\bar{\rho}$ is taken to be the average of ρ between streamline 2 and the wall.

We introduce mass flux correction factor, γ, by

$$\gamma = \frac{1}{u_2 z_2} \int_0^{z_2} u \, dz \tag{B-1}$$

In terms of γ we can write

$$\bar{\psi_2} = \gamma \bar{\rho} u_2 z_2 \tag{B-2}$$

Equation (B-2) is used at the beginning to calculate $\bar{\psi_2}$ and later on to calculate z_2 for rectangular ducts. For circular ducts r_1 is calculated by

$$r_1 = r_2 + \bar{\psi_2}/\gamma u_2 z_2 \tag{B-3}$$

From their definitions we have for the correction factors α and δ,

$$\alpha \psi_2 \bar{u}_2 = \int_0^{z_2} \rho u^2 dz \,, \quad \text{that is}$$

$$\alpha = \frac{1}{\gamma u_2^2 z_2} \int_0^{z_2} u^2 dz \,, \tag{B-4}$$

and similarly,

$$\delta = \frac{1}{\gamma u_2^3 z_2} \int_0^{z_3} u^3 dz \tag{B-5}$$

If one assumes as is often done, that the variation of h in the near wall region is similar to u then the correction factor β will be obviously equal to α. If some other variation of h is assumed then the integral

$$\beta = \frac{1}{\gamma u_2 z_2 h_2} \int_0^{z_2} hudz$$

evaluated with the assumed variation of h will yield β.

The velocity distribution in the near wall region, needed to evaluate the proceeding integrals, is calculated from the relation

$$\mu_t \frac{du}{dz} = \tau_w \tag{B-6}$$

where τ_w is the shear stress at the wall and μ_t is the viscosity including the turbulent contributions. To proceed further one must select some expression for μ_t. To illustrate the procedure we use von Driest's relation [8]

for μ_t (this relation has been successfully used in our own computations [7] and by several other authors [3,4]). For smooth walls we have from [8]

$$\mu_t = \mu + \rho k^2 z^2 \left[1 - \exp(-z\sqrt{\tau_w \rho}/\mu A)\right]^2 \frac{du}{dz} \tag{B-7}$$

where μ is the laminar viscosity and k and A are constants. Introducing non-dimensional variables

$$z_* = z\sqrt{\tau_w \rho}/\mu \ , \quad \text{and} \quad u_* = \rho u/\sqrt{\tau_w \rho} \tag{B-8}$$

we obtain on integrating (B-6) using (B-7)

$$u_* = \int_0^{z_*} \frac{2dz_*}{1 + \left[1 + 4k^2 z_*^2 \{1 - \exp(-z_*/A)\}^2\right]^{\frac{1}{2}}} \tag{B-9}$$

In terms of the non-dimensional variables integrals (B-1), (B-4), and (B-5) become, omitting the subscript 2,

$$\gamma = \frac{1}{u_* z_*} \int_0^{z_*} u_* dz_* \tag{B-10}$$

$$\alpha = \frac{1}{\gamma u_*^2 z_*} \int_0^{z_*} u_*^2 dz_* \ , \quad \text{and} \tag{B-11}$$

$$\delta = \frac{1}{\gamma u_*^3 z_*} \int_0^{z_*} u_*^3 dz_* \tag{B-12}$$

Introducting friction factor, f, and Reynolds number, Re, we have in terms of non-dimensional variables

$$f = \frac{\tau_w}{\rho u^2} = \frac{1}{u_*^2} , \quad \text{and} \tag{B-13}$$

$$Re = \frac{\rho u z}{\mu} = u_* z_* . \tag{B-14}$$

After having calculated the velocity distribution from (B-9) γ, α, δ, and f can be next evaluated as functions of Re once and for all and the results then later used during the flow field computations. We have numerically integrated these integrals using $k = 0.4$ and $A = 26$, the values recommended by Van Driest [8] for smooth walls. The results of the calculations are shown in Figs. (B-1) and (B-2). The numerically obtained relation between any of the factors and Re can now be stored for later use either by fitting a high order polynomial over the whole range of Re of interest or by breaking the range of Re into small segments and then approximating the relation over each segment by a straight line. The quantity V_1 introduced in section II in Eq. (6) is in terms of f given by

$$V_1 = f \frac{\rho_1 + \rho_2}{2} u_2 \tag{B-15}$$

and V_1^+ is given similarly with the quantities in the right hand side of (B-15) evaluated at $x + \Delta x$.

REFERENCES

[1] H. Schlichting, Boundary-Layer Theory (McGraw-Hill, New York, 1968).

[2] P.J. Roache, Computational Fluid Dynamics (Hermosa Publishers, Albuquerque, NM, 1972).

[3] S.V. Pantankar and D.B. Spalding, Heat and Mass Transfer in Boundary Layers (Margon-Grampian, London, 1967).

[4] E.D. Doss, Subsonic MHD-Diffuser Performance with High Blockage, Journ. of Energy 1 (1977), pp. 370-375.

[5] R.A. Gentry, R.E. Martin and R.J. Daly, An Eulerian Differencing Method for Unsteady Compressible Flow Problems, Journ. of Comp. Phys. 1 (1966), pp. 87-117.

[6] P. Fritzer, L.L. Lengyel and K.J. Witte, Self Consistent Calculation of the Gasdynamic and Electrical Properties of a Two-Dimensional MHD Flow, in Proceedings of the 12th Symposium on Engineering Aspects of MHD (1972), pp. III.7.1-III.7.8.

[7] M.S. Greywall and C.C.P. Pian, Velocity, Temperature, and Conductivity Profiles in Hydrogen-Oxygen MHD Duct Flows, to appear in Symposium on Fluids Engineering in Advanced Energy Conversion Systems, 1978 ASME Winter Annual Meeting (1978).

[8] E.R. van Driest, On Turbulent Flow Near a Wall, Journ. of the Aero. Sciences 23 (1956), pp. 1011-1011, 1036.

ACKNOWLEDGEMENTS

We would like to thank Dr. J.M. Smith and Dr. C.C.P. Pian of NASA-Lewis Research Center for many helpful discussions during the course of these investigations.

The work was partly supported by the National Aeronautics and Space Administration Grant NSG-3186 and partly by NASA-ASEE summer faculty program.

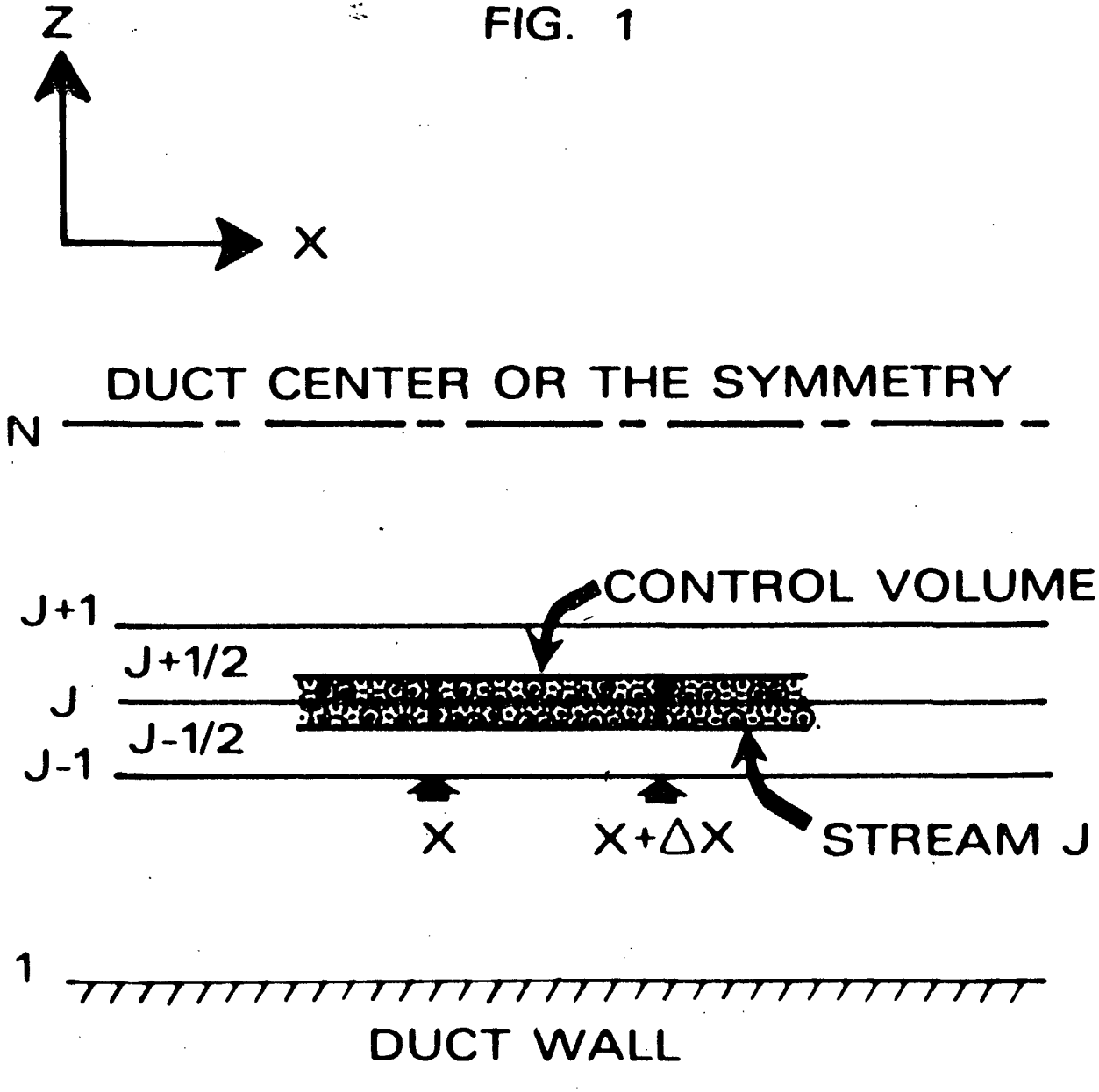
LIST OF FIGURES

Fig. 1. Partitioning of the flow in the duct into streams.

Fig. B-1. Variation of friction factor f with Reynolds number.

Fig. B-2. Variations of mass, momentum, and kinetic energy correction factors γ, α, and δ with Reynolds number.

FIG. 1


Z
X
DUCT CENTER OR THE SYMMETRY
N
CONTROL VOLUME
J+1
J+1/2
J
J-1/2
J-1
X
X+ΔX
STREAM J
1
DUCT WALL

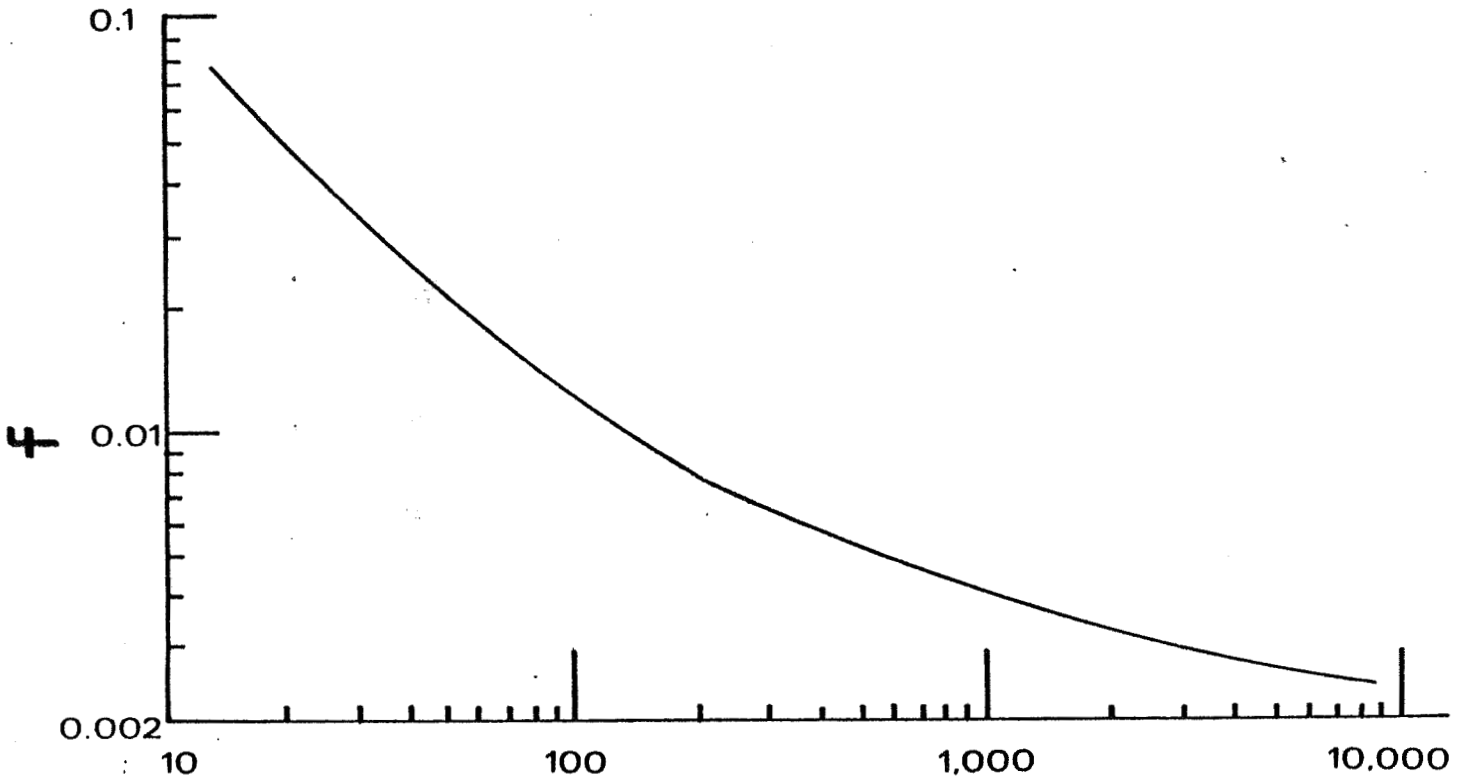


FIG. B-1

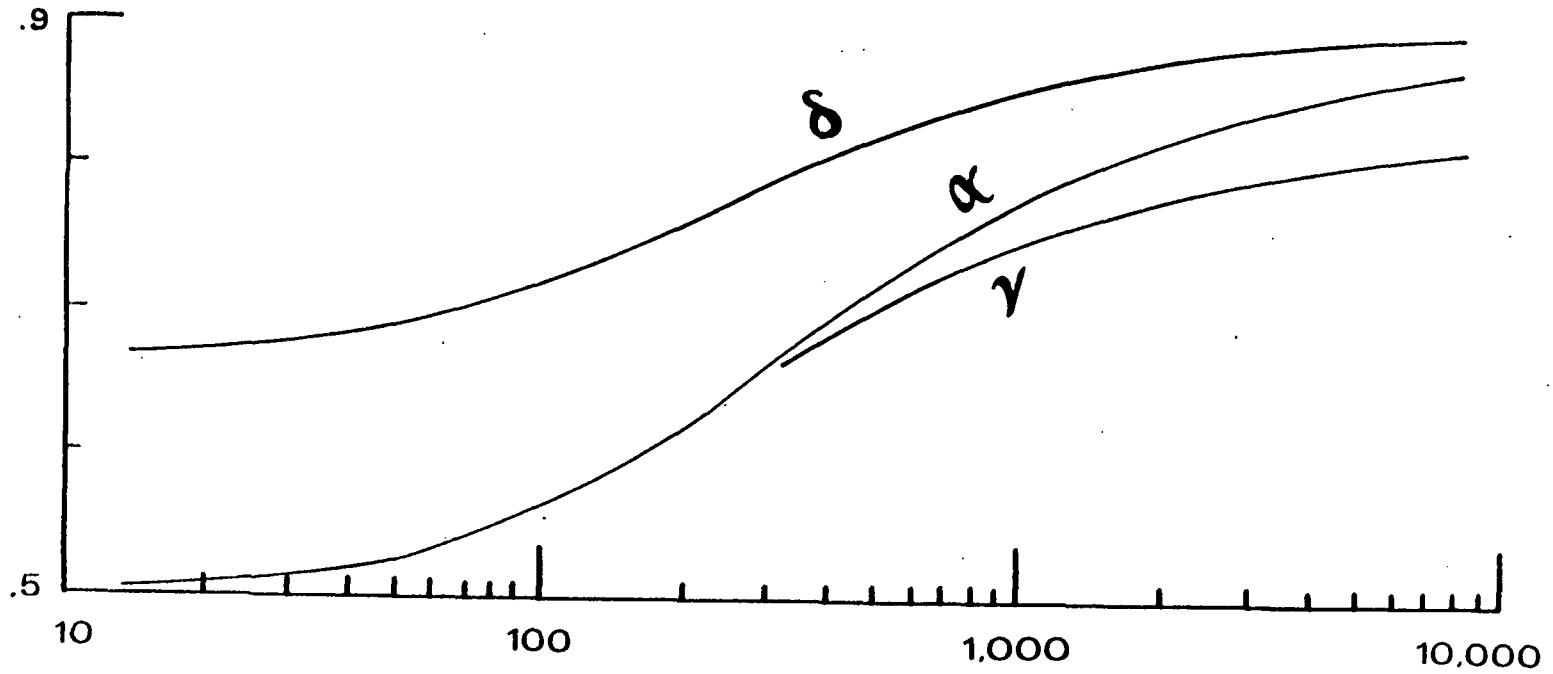


FIG. B-2